

SOME FOUR-DIMENSIONAL ALMOST HYPERCOMPLEX PSEUDO-HERMITIAN MANIFOLDS

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In this paper, a lot of examples of four-dimensional manifolds with an almost hypercomplex pseudo-Hermitian structure are constructed in several explicit ways. The received 4-manifolds are characterized by their linear invariants in the known aspects.

Introduction

In the study of almost hypercomplex manifolds the Hermitian metrics are well known. The parallel study of almost hypercomplex manifolds with skew-Hermitian metrics is in progress of development^{6, 7}.

Let (M, H) be an almost hypercomplex manifold, i.e. M is a $4n$ -dimensional differentiable manifold and H is a triple (J_1, J_2, J_3) of anticommuting almost complex structures, where $J_3 = J_1 \circ J_2$ ^{8, 2}.

A standard hypercomplex structure for all $x(x^i, y^i, u^i, v^i) \in T_p M$, $p \in M$ is defined in⁸ as follows

$$J_1 x(-y^i, x^i, v^i, -u^i), \quad J_2 x(-u^i, -v^i, x^i, y^i), \quad J_3 x(v^i, -u^i, y^i, -x^i). \quad (1)$$

Let us equip (M, H) with a pseudo-Riemannian metric g of signature $(2n, 2n)$ so that

$$g(\cdot, \cdot) = g(J_1 \cdot, J_1 \cdot) = -g(J_2 \cdot, J_2 \cdot) = -g(J_3 \cdot, J_3 \cdot). \quad (2)$$

We called such metric a *pseudo-Hermitian metric* on an almost hypercomplex manifold⁶. It generates a Kähler 2-form Φ and two pseudo-Hermitian metrics g_2 and g_3 by the following way

$$\Phi := g(J_1 \cdot, \cdot), \quad g_2 := g(J_2 \cdot, \cdot), \quad g_3 := g(J_3 \cdot, \cdot). \quad (3)$$

The metric g (g_2, g_3 , respectively) has an Hermitian compatibility with respect to J_1 (J_3, J_2 , respectively) and a skew-Hermitian compatibility with respect to J_2 and J_3 (J_1 and J_2, J_1 and J_3 , respectively).

On the other hand, a quaternionic inner product $\langle \cdot, \cdot \rangle$ in \mathbb{H} generates in a natural way the bilinear forms g, Φ, g_2 and g_3 by the following decomposition: $\langle \cdot, \cdot \rangle = -g + i\Phi + jg_2 + kg_3$.

The structure $(H, G) := (J_1, J_2, J_3; g, \Phi, g_2, g_3)$ is called a *hypercomplex pseudo-Hermitian structure* on M^{4n} or shortly a (H, G) -structure on M^{4n} . The manifold (M, H, G) is called an *almost hypercomplex pseudo-Hermitian manifold* or shortly an *almost (H, G) -manifold* ⁶.

The basic purpose of the recent paper is to construct explicit examples of the (H, G) -manifolds of the lowest dimension at $n = 1$ and to characterize them.

The following structural $(0, 3)$ -tensors play basic role for the characterization of the almost (H, G) -manifold

$$F_\alpha(x, y, z) = g((\nabla_x J_\alpha) y, z) = (\nabla_x g_\alpha)(y, z), \quad \alpha = 1, 2, 3,$$

where ∇ is the Levi-Civita connection generated by g .

It is well known ², that the almost hypercomplex structure $H = (J_\alpha)$ is a hypercomplex structure if the Nijenhuis tensors

$$N_\alpha(X, Y) = [X, Y] + J_\alpha [X, J_\alpha Y] + J_\alpha [J_\alpha X, Y] - [J_\alpha X, J_\alpha Y]$$

vanish for each $\alpha = 1, 2, 3$. Moreover, one H is hypercomplex iff two of N_α are zero.

Since g is a Hermitian metric with respect to J_1 , we use the classification of the almost Hermitian manifolds given in ⁵. According to it the basic class of these manifolds of dimension 4 are the class of almost Kähler manifolds $\mathcal{AK} = \mathcal{W}_2$ and the class of Hermitian manifolds $\mathcal{H} = \mathcal{W}_4$. The class of the \mathcal{AK} -manifolds are defined by condition $d\Phi = 0$ or equivalently $\sum_{x,y,z} F_1(x, y, z) = 0$. The class of the Hermitian 4-manifolds is determined by $N_1 = 0$ or

$$F_1(x, y, z) = \frac{1}{2} [g(x, y)\theta_1(z) - g(x, z)\theta_1(y) - g(x, J_1 y)\theta_1(J_1 z) + g(x, J_1 z)\theta_1(J_1 y)]$$

where $\theta_1(\cdot) = g^{ij} F_1(e_i, e_j, \cdot) = \delta\Phi(\cdot)$ for any basis $\{e_i\}_{i=1}^4$, and δ – the code-rivative.

On other side, the metric g is a skew-Hermitian one with respect to J_2 and J_3 . A classification of all almost complex manifolds with skew-Hermitian

metric (Norden metric or B-metric) is given in ³. The basic classes are:

$$\begin{aligned} \mathcal{W}_1 : F_\alpha(x, y, z) &= \frac{1}{4} [g(x, y)\theta_\alpha(z) + g(x, z)\theta_\alpha(y) \\ &\quad + g(x, J_\alpha y)\theta_\alpha(J_\alpha z) + g(x, J_\alpha z)\theta_\alpha(J_\alpha y)], \\ \mathcal{W}_2 : \bigcirc_{x, y, z} F_\alpha(x, y, J_\alpha z) &= 0, \quad \mathcal{W}_3 : \bigcirc_{x, y, z} F_\alpha(x, y, z) = 0, \end{aligned}$$

where $\theta_\alpha(\cdot) = g^{ij}F_\alpha(e_i, e_j, \cdot)$, $\alpha = 2, 3$, for an arbitrary basis $\{e_i\}_{i=1}^4$.

We denote the main subclasses of the respective complex manifolds by $\mathcal{W}(J_\alpha)$, where $\mathcal{W}(J_1) := \mathcal{W}_4(J_1)$ ⁵, and $\mathcal{W}(J_\alpha) := \mathcal{W}_1(J_\alpha)$ for $\alpha = 2, 3$ ³.

In the end of this section we recall some known facts from ⁶ and ⁷.

A sufficient condition an almost (H, G) -manifold to be an integrable one is following

Theorem 0.1 *Let (M, H, G) belongs to $\mathcal{W}(J_\alpha) \cap \mathcal{W}(J_\beta)$. Then (M, H, G) is of class $\mathcal{W}(J_\gamma)$ for all cyclic permutations (α, β, γ) of $(1, 2, 3)$.*

A pseudo-Hermitian manifold is called a *pseudo-hyper-Kähler manifold* (denotation $(M, H, G) \in \mathcal{K}$), if $F_\alpha = 0$ for every $\alpha = 1, 2, 3$, i.e. the manifold is Kählerian with respect to each J_α (denotation $(M, H, G) \in \mathcal{K}(J_\alpha)$).

Theorem 0.2 *If $(M, H, G) \in \mathcal{K}(J_\alpha) \cap \mathcal{W}(J_\beta)$ ($\alpha \neq \beta \in \{1, 2, 3\}$) then $(M, H, G) \in \mathcal{K}$.*

As g is an indefinite metric, there exists isotropic vector fields X on M . Following ⁴ we consider the invariants

$$||\nabla J_\alpha||^2 = g^{ij}g^{kl}g((\nabla_{e_i}J_\alpha)e_k, (\nabla_{e_j}J_\alpha)e_l), \quad \alpha = 1, 2, 3,$$

where $\{e_i\}_{i=1}^4$ is an arbitrary basis of T_pM , $p \in M$.

Definition 0.1 *An (H, G) -manifold is called: (i) isotropic Kählerian with respect to J_α if $||\nabla J_\alpha||^2 = 0$ for some $\alpha \in \{1, 2, 3\}$; (ii) isotropic hyper-Kählerian if it is isotropic Kählerian with respect to every J_α of H .*

Theorem 0.3 *Let M be an (H, G) -manifold of class $\mathcal{W} = \bigcap_\alpha \mathcal{W}(J_\alpha)$ ($\alpha = 1, 2, 3$) and $||\nabla J_\alpha||^2$ vanishes for some $\alpha = 1, 2, 3$. Then (M, H, G) is an isotropic hyper-Kähler manifold, but it is not pseudo-hyper-Kählerian in general.*

A geometric characteristic of the pseudo-hyper-Kähler manifolds according to the curvature tensor $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ induced by the Levi-Civita connection is given in ⁷.

Theorem 0.4 *Each pseudo-hyper-Kähler manifold is a flat pseudo-Riemannian manifold with signature $(2n, 2n)$.*

1 The two known examples of almost (H, G) -manifolds

1.1 A pseudo-Riemannian spherical manifold with (H, G) -structure

Following ¹⁰ we have considered in ⁶ and ⁷ a pseudo-Riemannian spherical manifold S_2^4 in pseudo-Euclidean vector space \mathbb{R}_2^5 of type $(- - + + +)$. The structure H is introduced on $\tilde{S}_2^4 = S_2^4 \setminus \{(0, 0, 0, 0, \pm 1)\}$ as in (1) and the pseudo-Riemannian metric g is the restriction of the inner product of \mathbb{R}_2^5 on \tilde{S}_2^4 . Therefore \tilde{S}_2^4 admits an almost hypercomplex pseudo-Hermitian structure. The corresponding manifold is of the class $\mathcal{W}(J_1)$ but it does not belong to \mathcal{W} and it has a constant sectional curvature $k = 1$. Moreover, we established that the considered manifold is conformally equivalent to a flat $\mathcal{K}(J_1)$ -manifold, which is not a \mathcal{K} -manifold and (\tilde{S}_2^4, H, G) is an Einstein manifold.

1.2 The Thurston manifold with (H, G) -structure

In ⁶ we have followed the interpretation of Abbena ¹ of the Thurston manifold. We have considered a 4-dimensional compact homogenous space L/Γ , where L is a connected Lie group and Γ is the discrete subgroup of L consisting of all matrices whose entries are integers. We have introduced the almost hypercomplex structure $H = (J_\alpha)$ on $T_E L$ as in (1) and we translate it on $T_A L$, $A \in L$, by the action of the left invariant vector fields. The J_α are invariant under the action of Γ , too. By analogy we have defined a left invariant pseudo-Riemannian inner product in $T_E L$. It generates a pseudo-Riemannian metric g on $M^4 = L$. Then the generated 4-manifold M is equipped with a suitable (H, G) -structure and (M, H, G) is a $\mathcal{W}(J_1)$ -manifold but it does not belong to the class \mathcal{W} .

2 Engel manifolds with almost (H, G) -structure

In the next two examples we consider $M = \mathbb{R}^4 = \{(x^1, x^2, x^3, x^4)\}$ with a basis $\{e_1 = \frac{\partial}{\partial x^1}, e_2 = \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}, e_3 = -\frac{\partial}{\partial x^3}, e_4 = -\frac{\partial}{\partial x^4}\}$ and an Engel structure $\mathcal{D} = \text{span}\{e_1, e_2\}$, i.e. an absolutely non-integrable regular two-dimensional distribution on TM ⁴.

2.1 Double isotropic hyper-Kählerian structures but neither hypercomplex nor symplectic

At first we use the introduced there a pseudo-Riemannian metric and almost complex structures given by

$$\begin{aligned} g &= (dx^1)^2 + \{1 - (x^1)^2 - (x^3)^2\}(dx^2)^2 - (dx^3)^2 \\ &\quad - (dx^4)^2 - 2x^1 dx^2 dx^3 + 2x^3 dx^2 dx^4, \\ J &: J e_1 = e_2, J e_2 = -e_1, J e_3 = e_4, J e_4 = -e_3, \\ J' &: J' e_1 = e_2, J' e_2 = -e_1, J' e_3 = -e_4, J' e_4 = e_3. \end{aligned} \quad (4)$$

It is given in ⁴ that (J, g) and (J', g) are a pair of indefinite almost Hermitian structures which are isotropic Kähler but neither complex nor symplectic.

It is clear that $\{e_i\}_{i=1}^4$ is an orthonormal $(++--)$ -basis. We accomplish the introduction of an (H, G) -structure on M by

$$J_1 := J'; J_2 : J_2 e_1 = e_3, J_2 e_2 = e_4, J_2 e_3 = -e_1, J_2 e_4 = -e_2; J_3 := J_1 J_2.$$

By direct computations we verify that the constructed manifold is an (H, G) -manifold and it is isotropic hyper-Kählerian but not Kählerian and not integrable with non-vanishing Lie forms with respect to any J_α ($\alpha = 1, 2, 3$).

Remark. If we define J_1 as J instead of J' then the kind of example is not changed. So we receive a pair of almost (H, G) -structures corresponding to the given almost complex structures.

The non-zero components of the curvature tensor R and the basic linear invariant of the almost Hermitian manifold (M, J_1, g) are given in ⁴ by

$$\begin{aligned} R_{1221} &= \frac{3}{4}, R_{1331} = -R_{2142} = -R_{2442} = -R_{3143} = R_{3443} = \frac{1}{4}, R_{2332} = 1; \\ \|F_1\|^2 &= 0, \|N_1\|^2 = 8, \tau = 0, \tau_1^* = -2, \end{aligned}$$

where the following denotations are used for $\varepsilon_a = \|e_a\|^2$

$$\begin{aligned} \|F_1\|^2 &= \|\nabla\Phi\|^2 = \sum_{a,b,c=1}^4 \varepsilon_a \varepsilon_b \varepsilon_c F_1(e_a, e_b, e_c)^2, \\ \|N_1\|^2 &= \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b \|N_1(e_a, e_b)\|^2, \\ \tau &= \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b R(e_a, e_b, e_b, e_a), \quad \tau_1^* = \frac{1}{2} \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b R(e_a, J_1 e_a, e_b, J_1 e_b). \end{aligned}$$

We get the corresponding linear invariants with respect to J_2 and J_3 :

$$\begin{aligned} \|F_2\|^2 &= 0, & \|N_2\|^2 &= 0, & \tau_2^* &= 0; \\ \|F_3\|^2 &= 0, & \|N_3\|^2 &= -8, & \tau_3^* &= 0, \end{aligned}$$

where $\tau_\alpha^* = \sum_{a,b=1}^4 \varepsilon_a \varepsilon_b R(e_a, e_b, J_\alpha e_b, e_a)$; $\alpha = 2, 3$.

2.2 Double isotropic hyper-Kählerian structures which are non-integrable but symplectic

Now we consider the same Engel manifold $(M = \mathbb{R}^4, \mathcal{D})$ but let the pseudo-Riemannian metric and the pair of almost complex structures be defined by other way: ⁴

$$\begin{aligned} g &= (dx^1)^2 - \{1 - (x^1)^2 + (x^3)^2\}(dx^2)^2 + (dx^3)^2 \\ &\quad - (dx^4)^2 - 2x^1 dx^2 dx^3 + 2x^3 dx^2 dx^4, \\ J &: J e_1 = e_3, J e_2 = e_4, J e_3 = -e_1, J e_4 = -e_2, \\ J' &: J' e_1 = e_3, J' e_2 = -e_4, J' e_3 = -e_1, J' e_4 = e_2. \end{aligned}$$

In this case $\{e_i\}_{i=1}^4$ is an orthonormal basis of type $(+ - + -)$. It is shown that (M, J, g) and (M, J', g) are a pair of isotropic Kähler almost Kähler manifolds with vanishing linear invariants.

We accomplish the introduced almost complex structures to almost hypercomplex structures on M by using the following way: we set the given J (resp. J') as J_1 (resp. J'_1), then we introduce J_2 (resp. J'_2) by

$$\begin{aligned} J_2 &: J_2 e_1 = e_2, J_2 e_2 = -e_1, J_2 e_3 = -e_4, J_2 e_4 = e_3; \\ J'_2 &: J'_2 e_1 = e_2, J'_2 e_2 = -e_1, J'_2 e_3 = e_4, J'_2 e_4 = -e_3 \end{aligned} \quad (5)$$

and finally we set $J_3 := J_1 J_2$ (resp. $J'_3 := J'_1 J'_2$).

It is easy to check that $H = (J_\alpha)$ and $H' = (J'_\alpha)$ together with g generate a pair of almost hypercomplex pseudo-Hermitian structures on M .

We characterize the both received (H, G) -manifolds as isotropic hyper-Kähler but not Kähler manifolds and not integrable manifolds with non-vanishing Lie forms with respect to any J_α . Moreover, they have the following linear invariants:

$$\|N_1\|^2 = 0, \quad \|N_2\|^2 = -\|N_3\|^2 = 8, \quad \|F_\alpha\|^2 = 0, \quad \tau = \tau_\alpha^* = 0 \quad (\alpha = 1, 2, 3).$$

3 Real spaces with almost (H, G) -structure

3.1 Real semi-space with almost (H, G) -structure

Let us consider the real semi-space $\mathbb{R}_+^4 = \{(x^1, x^2, x^3, x^4), x^i \in \mathbb{R}, x^1 > 0\}$ with the basis given by $\{e_1 = x^1 \frac{\partial}{\partial x^1}, e_2 = x^1 \frac{\partial}{\partial x^2}, e_3 = x^1 \frac{\partial}{\partial x^3}, e_4 = x^1 \frac{\partial}{\partial x^4}\}$. It is clear that this basis is orthonormal of type $(+ + - -)$ with respect to the pseudo-Riemannian metric $g = \{(dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2\} / (x^1)^2$.

We introduce an almost hypercomplex structure $H = (J_\alpha)$ as follows

$$\begin{aligned} J_1 : J_1 e_1 &= e_2, J_1 e_2 = -e_1, J_1 e_3 = e_4, J_1 e_4 = -e_3; \\ J_2 : J_2 e_1 &= e_3, J_2 e_2 = -e_4, J_2 e_3 = -e_1, J_2 e_4 = e_2; \quad J_3 = J_1 J_2 \end{aligned} \quad (6)$$

and we check that H and g generates an almost (H, G) -structure on \mathbb{R}_+^4 .

We verify immediately that H is integrable and the obtained hypercomplex pseudo-Hermitian manifold (\mathbb{R}_+^4, H, G) belongs to the class $\mathcal{W} = \bigcap_\alpha \mathcal{W}(J_\alpha)$ but it is not isotropic Kählerian with respect to J_α ($\alpha = 1, 2, 3$).

By direct computations we obtain for the curvature tensor that $R = -\pi_1$, i.e. the manifold has constant sectional curvatures $k = -1$ and it is an Einstein manifold. Moreover, the linear invariants are

$$\begin{aligned} \|N_\alpha\|^2 &= 0, \quad 2\|F_1\|^2 = 4\|\theta_1\|^2 = -\|F_\beta\|^2 = -\|\theta_\beta\|^2 = 16, \\ \tau &= -3\tau_1^* = -12, \quad \tau_\beta^* = 0, \end{aligned}$$

where $\alpha = 1, 2, 3$; $\beta = 2, 3$; and (\mathbb{R}_+^4, H, G) is conformally equivalent to a pseudo-hyper-Kähler manifold by the change $\bar{g} = (x^1)^2 g$.

3.2 Real quarter-space with almost (H, G) -structure

Let the real quarter-space

$$M = \mathbb{R}_+^2 \times \mathbb{R}_-^2 = \{(x^1, x^2, x^3, x^4), x^i \in \mathbb{R}, x^1 > 0, x^3 > 0\}$$

be equipped with a pseudo-Riemannian metric

$$g = \frac{1}{(x^1)^2} \{(dx^1)^2 + (dx^2)^2\} - \frac{1}{(x^3)^2} \{(dx^3)^2 + (dx^4)^2\}.$$

Then the basis $\{e_1 = x^1 \frac{\partial}{\partial x^1}, e_2 = x^1 \frac{\partial}{\partial x^2}, e_3 = x^3 \frac{\partial}{\partial x^3}, e_4 = x^3 \frac{\partial}{\partial x^4}\}$ is an orthonormal one of type $(++--)$. We introduce an almost hypercomplex structure $H = (J_\alpha)$ ($\alpha = 1, 2, 3$) as in the previous example by (6).

The received almost (H, G) -manifold is a $\mathcal{K}(J_1)$ -manifold and an isotropic hyper-Kähler manifold. As a corollary, $N_1 = 0$, $F_1 = 0$, $\theta_1 = 0$ and hence $\|N_1\|^2 = \|F_1\|^2 = \|\theta_1\|^2 = 0$. For the J_α ($\alpha = 2, 3$) the Nijenhuis tensors N_α , the tensors F_α , and the Lie forms θ_α are non-zero (therefore H is not integrable), but the linear invariants $\|N_\alpha\|^2$, $\|F_\alpha\|^2$ and $\|\theta_\alpha\|^2$ vanish.

The non-zero components of the curvature tensor are given by $R_{1221} = -R_{3443} = -1$. For the Ricci tensor we have $\rho_{ii} = -1$ ($i = 1, \dots, 4$). Therefore the basic non-zero sectional curvatures are $k(e_1, e_2) = -k(e_3, e_4) = -1$ and the scalar curvatures τ , τ_α^* ($\alpha = 1, 2, 3$) are zero.

4 Real pseudo-hyper-cylinder with almost (H, G) -structure

Let \mathbb{R}_2^5 be a pseudo-Euclidean real space with an inner product $\langle \cdot, \cdot \rangle$ of signature $(+++--)$. Let us consider a pseudo-hyper-cylinder defined by

$$S : (z^2)^2 + (z^3)^2 - (z^4)^2 - (z^5)^2 = 1,$$

where $Z(z^1, z^2, z^3, z^4, z^5)$ is the positional vector at $p \in S$. We use the following parametrization of S in the local coordinates (u^1, u^2, u^3, u^4) of p :

$$Z = Z(u^1, \cosh u^4 \cos u^2, \cosh u^4 \sin u^2, \sinh u^4 \cos u^3, \sinh u^4 \sin u^3).$$

We consider a manifold on the surface $\tilde{S} = S \setminus \{u^4 = 0\}$. Then the basis $\{e_1 = \partial_1, e_2 = \frac{1}{\cosh u^4} \partial_2, e_3 = \frac{1}{\sinh u^4} \partial_3, e_4 = \partial_4\}$ of $T_p \tilde{S}$ at $p \in \tilde{S}$ is an orthonormal basis of type $(++--)$ with respect to the restriction g of $\langle \cdot, \cdot \rangle$ on \tilde{S} . Here and further ∂_i denotes $\frac{\partial Z}{\partial u^i}$ for $i = 1, \dots, 4$;

We introduce an almost hypercomplex structure by the following way

$$\begin{aligned} J_1 : J_1 e_1 &= e_2, J_1 e_2 = -e_1, J_1 e_3 = -e_4, J_1 e_4 = e_3, \\ J_2 : J_2 e_1 &= e_3, J_2 e_2 = e_4, J_2 e_3 = -e_1, J_2 e_4 = -e_2; \quad J_3 = J_1 J_2 \end{aligned} \quad (7)$$

and check that $H = (J_\alpha)$ and the pseudo-Riemannian metric g generate an almost (H, G) -structure on \tilde{S} .

By straightforward calculations with respect to $\{e_i\}$ ($i = 1, \dots, 4$) we receive that the almost (H, G) -manifold \tilde{S} is not integrable with non-zero Lie forms regarding any J_α of H and it has the following linear invariants:

$$\begin{aligned} \|N_1\|^2 &= 2\|F_1\|^2 = 2\|\nabla J_1\|^2 = 8\|\theta_1\|^2 = -8 \tanh^2 u^4; \\ \|N_2\|^2 &= -8 \coth^2 u^4, \quad \|\theta_2\|^2 = (2 \tanh u^4 + \coth u^4)^2, \\ \|F_2\|^2 &= \|\nabla J_2\|^2 = 4(2 \tanh^2 u^4 + \coth^2 u^4); \\ \|N_3\|^2 &= -8(\tanh u^4 - \coth u^4)^2, \quad \|\theta_3\|^2 = (\tanh u^4 + \coth u^4)^2, \\ \|F_3\|^2 &= \|\nabla J_3\|^2 = 4(\tanh^2 u^4 + \coth^2 u^4). \end{aligned}$$

The non-zero components of the curvature tensor and the corresponding Ricci tensor and scalar curvatures are given by

$$\begin{aligned} R_{2332} &= -1, \quad R_{2442} = -\tanh^2 u^4, \quad R_{3443} = \coth^2 u^4 \\ \rho_{22} &= 1 + \tanh^2 u^4, \quad \rho_{33} = -1 - \coth^2 u^4, \quad \rho_{44} = -\tanh^2 u^4 - \coth^2 u^4 \\ \tau &= 2(1 + \tanh^2 u^4 + \coth^2 u^4), \quad \tau_\alpha^* = 0, \quad \alpha = 1, 2, 3. \end{aligned}$$

Hence (\tilde{S}, H, G) has zero associated scalar curvatures and H is a non-integrable structure on it.

5 Complex surfaces with almost (H, G) -structure

The following three examples concern several surfaces $S_{\mathbb{C}}^2$ in a 3-dimensional complex Euclidean space $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$. It is well known that the decomplexification of \mathbb{C}^3 to \mathbb{R}^6 using the i -splitting, i.e. $(Z^1, Z^2, Z^3) \in \mathbb{C}^3$, where $Z^k = x^k + iy^k$ ($x^k, y^k \in \mathbb{R}$), is identified with $(x^1, x^2, x^3, y^1, y^2, y^3) \in \mathbb{R}^6$. Then the multiplying by i in \mathbb{C}^3 induces the standard complex structure J_0 in \mathbb{R}^6 . The real and the opposite imaginary parts of the complex Euclidean inner product $\Re\langle \cdot, \cdot \rangle$ and $-\Im\langle \cdot, \cdot \rangle$ are the standard skew-Hermitian metrics g_0 and $\tilde{g}_0 = g_0(\cdot, J_0 \cdot)$ in $(\mathbb{R}^6, J_0, g_0, \tilde{g}_0)$, respectively. So, the natural decomplexification of an n -dimensional complex Euclidean space is the $2n$ -dimensional real space with a complex skew-Hermitian structure (J_0, g_0, \tilde{g}_0) .

5.1 Complex cylinder with almost (H, G) -structure

Let $S_{\mathbb{C}}^2$ be the cylinder in $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$ defined by $(Z^1)^2 + (Z^2)^2 = 1$. Let the corresponding surface S^4 in $(\mathbb{R}^6, J_0, g_0, \tilde{g}_0)$ be parameterized as follows

$$S^4 : Z = Z(\cos u^1 \cosh u^3, \sin u^1 \cosh u^3, u^2, \sin u^1 \sinh u^3, -\cos u^1 \sinh u^3, u^4).$$

Then the local basis $\{\partial_1, \dots, \partial_4\}$ is orthonormal of type $(++--)$ and it generates the metric $g = (du^1)^2 + (du^2)^2 - (du^3)^2 - (du^4)^2$ on S^4 . The almost hypercomplex structure H is determined as usually by (1). It is easy to verify that the received (H, G) -manifold is a flat pseudo-hyper-Kähler manifold.

5.2 Complex cone with almost (H, G) -structure

Now let $S_{\mathbb{C}}^2$ be the complex cone in $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$ determined by the equation $(Z^1)^2 + (Z^2)^2 - (Z^3)^2 = 0$. Then we consider the corresponding 4-dimensional surface S in $(\mathbb{R}^6, J_0, g_0, \tilde{g}_0)$ by the following parametrization of Z :

$$(u^1 \cos u^2 \cosh u^4 - u^3 \sin u^2 \sinh u^4, u^1 \sin u^2 \cosh u^4 + u^3 \cos u^2 \sinh u^4, u^1, u^1 \sin u^2 \sinh u^4 + u^3 \cos u^2 \cosh u^4, -u^1 \cos u^2 \sinh u^4 + u^3 \sin u^2 \cosh u^4, u^3).$$

Further we consider a manifold on $\tilde{S} = S \setminus \{0, 0, 0, 0, 0, 0\}$, i.e. we exclude the plane $u^1 = u^3 = 0$ from the domain of S which maps the origin. Then the derived metric g on \tilde{S} has the following non-zero components regarding $\{\partial_k\}$:

$$g_{11} = -g_{33} = 2, \quad g_{22} = -g_{44} = (u^1)^2 - (u^3)^2, \quad g_{24} = g_{42} = 2u^1 u^3.$$

We receive the following orthonormal basis of signature $(++--)$:

$$\left\{ e_1 = \frac{1}{\sqrt{2}}\partial_1, e_2 = \lambda\partial_2 + \mu\partial_4, e_3 = \frac{1}{\sqrt{2}}\partial_3, e_4 = -\mu\partial_2 + \lambda\partial_4 \right\},$$

where $\lambda = u^1/\{(u^1)^2 + (u^3)^2\}$, $\mu = u^3/\{(u^1)^2 + (u^3)^2\}$. We introduce a structure H as in (1). It is easy to check that H and g generate an almost (H, G) -structure on \tilde{S} . By direct computations we get that the received (H, G) -manifold is a flat hypercomplex manifold which is Kählerian with respect to J_1 but it does not belong to $\mathcal{W}(J_2)$ or $\mathcal{W}(J_3)$ and the Lie forms θ_2 and θ_3 are non-zero. The corresponding linear invariants are given by

$$\begin{aligned}\|F_2\|^2 &= \|\nabla J_2\|^2 = 2\|\theta_2\|^2 = 16\{\mu^2 - \lambda^2\}, \\ \|F_3\|^2 &= \|\nabla J_3\|^2 = 2\|\theta_3\|^2 = 4\{\mu^2 - \lambda^2\}.\end{aligned}$$

5.3 Complex sphere with almost (H, G) -structure

In this case let $S_{\mathbb{C}}^2$ be the unit sphere in $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$ defined by $(Z^1)^2 + (Z^2)^2 + (Z^3)^2 = 1$. After that we consider the corresponding 4-surface S in $(\mathbb{R}^6, J_0, g_0, \tilde{g}_0)$ with the following parametrization of $Z(x^1, x^2, x^3, y^1, y^2, y^3)$:

$$\begin{aligned}x^1 &= \cos u^1 \cos u^2 \cosh u^3 \cosh u^4 - \sin u^1 \sin u^2 \sinh u^3 \sinh u^4, \\ x^2 &= \cos u^1 \sin u^2 \cosh u^3 \cosh u^4 + \sin u^1 \cos u^2 \sinh u^3 \sinh u^4, \\ x^3 &= \sin u^1 \cosh u^3, \\ S: \quad y^1 &= \cos u^1 \sin u^2 \cosh u^3 \sinh u^4 + \sin u^1 \cos u^2 \sinh u^3 \cosh u^4, \\ y^2 &= -\cos u^1 \cos u^2 \cosh u^3 \sinh u^4 + \sin u^1 \sin u^2 \sinh u^3 \cosh u^4, \\ y^3 &= -\cos u^1 \sinh u^3.\end{aligned}$$

Further we consider a manifold on $\tilde{S} = S \setminus \{0, 0, \pm 1, 0, 0, 0\}$, i.e. we exclude the set $u^1 = \pm\pi/2, u^3 = 0$ from the domain $(-\pi, \pi)^2 \times \mathbb{R}^2$ of S which maps the pair of "poles".

The induced metric on \tilde{S} has the following non-zero local components:

$$\begin{aligned}g_{11} &= -g_{33} = 1, \quad g_{22} = -g_{44} = \cos^2 u^1 \cosh^2 u^3 - \sin^2 u^1 \sinh^2 u^3, \\ g_{24} &= g_{42} = 2 \sin u^1 \cos u^1 \sinh u^3 \cosh u^3.\end{aligned}$$

Further we use the following orthonormal basis of signature $(++--)$:

$$\{e_1 = \partial_1, e_2 = \lambda \partial_2 + \mu \partial_4, e_3 = \partial_3, e_4 = -\mu \partial_2 + \lambda \partial_4\},$$

where $\lambda = \frac{\cos u^1 \cosh u^3}{\cos^2 u^1 + \sinh^2 u^3}$, $\mu = \frac{\sin u^1 \sinh u^3}{\cos^2 u^1 + \sinh^2 u^3}$. We introduce a structure H as in (1) and we verify that H and g generate an almost (H, G) -structure on \tilde{S} . By direct computations we get that (\tilde{S}, H, G) is a $\mathcal{K}(J_2)$ -manifold of pointwise constant totally real sectional curvatures

$$\nu = \frac{\sinh^2 2u^3 - \sin^2 2u^1}{4(\cos^2 u^1 + \sinh^2 u^3)^4}, \quad \nu_2^* = \frac{\sin 2u^1 \sinh 2u^3}{2(\cos^2 u^1 + \sinh^2 u^3)^4},$$

where $\nu := \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}$, $\nu_2^* := \frac{R(x,y,y,J_2x)}{\pi_1(x,y,y,x)}$ for a basis $\{x, y\}$ of any non-degenerate totally real section σ (i.e. $\sigma \perp J_2\sigma$). (\tilde{S}, J_2, g) is an almost Einstein manifold since its Ricci tensor is $\rho = 2(\nu g - \nu_2^* g_2)$. But, the Nijenhuis tensors and the Lie forms corresponding to other two almost complex structures J_1 and J_3 are non-zero. Beside that, we receive the following linear invariants:

$$\begin{aligned} \tau &= 8\nu, & \tau_1^* &= 0, & \tau_2^* &= 8\nu_2^*, & \tau_3^* &= 0, \\ \|N_1\|^2 &= 2\|\nabla J_1\|^2 = 8\|\theta_1\|^2 = -32\nu, & -\|N_3\|^2 &= 2\|\nabla J_3\|^2 = 8\|\theta_3\|^2 = 32\nu. \end{aligned}$$

6 Lie groups with almost (H, G) -structure

The next two examples are inspired from an example of a locally flat almost Hermitian surface constructed in ⁹. Let \mathcal{L} be a connected Lie subgroup of $\mathcal{GL}(4, \mathbb{R})$ consisting of matrices with the following non-zero entries

$$\begin{aligned} a_{11} &= a_{22} = \cos u_1, & a_{12} &= -a_{21} = \sin u_1, \\ a_{13} &= u_2, & a_{23} &= u_3, & a_{33} &= 1, & a_{44} &= \exp u_4 \end{aligned}$$

for arbitrary $u^1, u^2, u^3, u^4 \in \mathbb{R}$.

The Lie algebra of \mathcal{L} is isomorphic to the Lie subalgebra of $\mathfrak{gl}(4; \mathbb{R})$ generated by the matrices X_1, X_2, X_3, X_4 with the the following non-zero entries:

$$(X_1)_{13} = (X_2)_{12} = -(X_2)_{21} = (X_3)_{23} = (X_4)_{44} = 1.$$

6.1 A Lie group as a complex manifold but non-hypercomplex one

For the first recent example let us substitute the following pseudo-Riemannian g for the metric on \mathcal{L} used there: $g(X_i, X_j) = \varepsilon_a \delta_{ij}$, where $1 \leq i, j \leq 4$; $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4 = 1$. Further we introduce an \mathcal{L} -invariant almost hypercomplex structure H on \mathcal{L} as in (1). Then, there is generated an almost (H, G) -structure on \mathcal{L} and the received manifold is complex with respect to J_2 but non-hypercomplex and the Lie forms do not vanish. The non-zero components of the curvature tensor R is determined by $R_{1221} = R_{1331} = -R_{2332} = 1$ and the linear invariants are the following:

$$\begin{aligned} \|N_1\|^2 &= 2\|\nabla J_1\|^2 = 8\|\theta_1\|^2 = -\|\nabla J_2\|^2 = -2\|\theta_2\|^2 = \|N_3\|^2 = -8, \\ \|\nabla J_3\|^2 &= 12\|\theta_3\|^2 = 12, & \tau &= -\tau_1^* = 2, & \tau_2^* &= \tau_3^* = 0. \end{aligned}$$

6.2 A Lie group as a flat Kähler manifold but non-hypercomplex one

For the second example we use the following pseudo-Riemannian g on \mathcal{L} : $g(X_i, X_j) = \varepsilon_a \delta_{ij}$, where $1 \leq i, j \leq 4$; $\varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = -\varepsilon_4 = 1$. We actually

substitute only the type of the signature: $(+ - + -)$ for $(+ + - -)$ of the basis $\{X_1, X_2, X_3, X_4\}$. Then we introduce H by the following different way:

$$\begin{aligned} J_1 X_1 &= X_3, & J_1 X_2 &= X_4, & J_1 X_3 &= -X_1, & J_1 X_4 &= -X_2, \\ J_2 X_1 &= -X_4, & J_2 X_2 &= X_3, & J_2 X_3 &= -X_2, & J_2 X_4 &= X_1, & J_3 &= J_1 J_2. \end{aligned}$$

Therefore we obtain that the constructed (H, G) -manifold is flat and Kählerian with respect to J_1 but regarding J_2 and J_3 it is not complex and the structural tensors have the form $F_2(X, Y, Z) = -\theta_2(J_3 X)g(Y, J_3 Z)$, $F_3(X, Y, Z) = -\theta_3(J_2)g(Y, J_2 Z)$. The non-zero linear invariants for $\beta = 2, 3$ are the following: $-\|N_\beta\|^2 = 2\|\nabla J_\beta\|^2 = 2\|F_\beta\|^2 = 8\|\theta_\beta\|^2 = 8$.

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